THEORY OF REGULAR ELECTROSTATIC BEAMS OF CHARGED PARTICLES

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In some papers concerned with the exact solution of the equations of a nonrelativistic single-energy beam of charged particles (e.g., [1, 2]), the opinion has been expressed that, while the method of separation of the variables has possibilities, serious difficulties can arise in obtaining the actual systems with separated variables. In particular, it has become popular, when investigating regular electrostatic flows, to transform to a coordinate system connected with the trajectory. In this system the velocity vector only has one component, say $\mathbf{V} = \{v_{x^1}, 0, 0\}$, so that flow only occurs in the x^1 direction (x^1 flow). We shall also refer to a single-component flow, as in [3]. This method is thought (e.g., [4]) to be effective for a wide class of flows. The question of the coordinate systems that allow flows in the x^1 direction is more specialized than the general problem of separation of variables.

The concept of an x^1 flow is discussed in §1 from the point of view of its utility for obtaining solutions of the regular electrostatic beam equations. Transformation to a coordinate system connected with the trajectory is found to be only justifiable for four orthogonal systems: cartesian, cylindrical, helical-cylindrical, and spherical. It is shown that, in the case of two-dimensional systems on a plane with conformal metric, the condition required for an x^1 flow and the conditions for the space to be euclidean can be used effectively to establish the existence in the given class of coordinate systems of an x^1 flow starting from a fictitious emitter (§2). The usual tensor notation is employed.

§1. Following [5], a flow will be called regular if the generalized momentum is a potential vector. In the absence of an external magnetic field, H = 0 (electrostatic beam), this is equivalent to having

$$e^{ikl}\frac{\partial v_i}{\partial x^k} = 0, \qquad v_i = \frac{\partial W}{\partial x^i}$$

where W is the action referred to the particle mass, and v_i are the covariant velocity components. A single-energy regular nonrelativistic beam of particles of like charge is described in the stationary case with H = 0 [3, 6] by a single nonlinear fourth order differential equation in W. In a curvilinear coordinate system x^i (i = 1, 2, 3), the metric in which is given by

$$dS^{(2)} = g_{ik}\, dx^i\, dx^k, \qquad (1.1)$$

the equation in question is

$$\frac{\partial}{\partial x^m}\left\{g^{mn}\frac{\partial W}{\partial x^n}\frac{\partial}{\partial x^j}\left[\sqrt{g}g^{jl}\frac{\partial}{\partial x^l}\left(g^{ik}\frac{\partial W}{\partial x^i}\frac{\partial W}{\partial x^k}\right)\right]\right\} = 0. \qquad (1.2)$$

In the case of a flow in the x^1 direction, this becomes [3, 7]

$$f(x)\,w^{1/2}\frac{d^2w}{(dx^1)^2} + \frac{\partial f(x)}{\partial x^1}w^{1/2}\frac{dw}{dx^1} + f(x)\,h(x)\,w^{1/2} = F(x^2, x^3),$$

$$f(x) = \left[\frac{g_{22}g_{33}}{(g_{11})^5}\right]^{1/2}, \qquad h(x) = \frac{(g_{11})^2}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ik}\frac{\partial g^{11}}{\partial x^k}\right),$$

$$w = (v_1)^2 = \left(\frac{dW}{\partial x^1}\right)^2, \qquad (1.3)$$

where $F(x^2, x^3)$ is a function resulting from integration with respect to x^1, and $f(x) = f(x^1, x^2, x^3)$.

Several authors [7–12] have considered the necessary and sufficient conditions for a single-component flow in the x^1 direction: when these are satisfied, (1.3) becomes an ordinary differential equation in $w(x^1)$. The sufficient conditions for x^1 flow are given in [7] as

$$f(x) = \Phi(x^1)F(x^2, x^3), \qquad h(x) = \Psi(x^1). \qquad (1.4)$$

Examples of solutions for which (1.4) are not satisfied are mentioned in [8, 9]. The example in [8] is plane flow along hyperbolic trajectories with constant space charge density, first discussed in [13], for which

$$f(x) = 4\,[(x^1)^2 + (x^2)^2], \qquad h(x) = [(x^1)^2 + (x^2)^2]^{-1},$$

$$x^1 = {}^1/_2\,(x^2 - y^2), \qquad x^2 = xy, \qquad W = x^1, \qquad (1.5)$$

where x, y are Cartesian coordinates. It is shown in [9] that for the plane periodic flow described in [14]

$$f(x) = 16\,[1 - 2\exp(x^2)\cos x^1 + \exp(2x^2)],$$

$$x^1 = \mathrm{Re}\,(2i \ln \mathrm{sc}\, z), \qquad x^2 = \mathrm{Im}\,(2i \ln \mathrm{sc}\, z),$$

$$W = x^1, \qquad z = x + iy. \qquad (1.6)$$

Solutions (1.5) and (1.6) correspond to flows which may not start from the actual emitter. In [9], a solution is quoted which defines space-charge-limited emission from the plane y = 0:

$$x^1 = e^{\alpha x}Y(y), \qquad x^2 = x - \int\frac{\alpha Y}{dY/dy}dy,$$

$$W = x^1 \quad (\alpha = \mathrm{const}), \qquad (1.7)$$

where Y is a function satisfying an ordinary differential equation [1]; the solution (1.7) is found by separation of the variables. Notice that (1.7), like other solutions of the form $W = K(x^1)L(x^2)$, is invariant [15]. The metric of system (1.7) again does not satisfy conditions (1.4).

It is shown in [16, 17] that when Eqs. (1.4) are satisfied single-component flows are only possible in four orthogonal coordinate systems: Cartesian x, y, z; cylindrical R, ψ, z; helical-cylindrical q_1, q_2, z; and and spherical r, θ, ψ. The class of coordinate systems considered in [17] is wider than in [16]. It was found that the trajectories could be straight lines, circles, or helices. Let us try to see why there should be so few possible trajectories: whether it is that conditions (1.4) are not sufficiently general, or for some other reason.

Undoubtedly, any regular flow can be represented as a single-component flow by taking the particle trajectory as one of the coordinate axes and putting $x^1 = W$. In this coordinate system the covariant velocity component is unity $v_1 = 1$ ($v_2 = v_3 = 0$), and from (1.3) the necessary condition for x^1 flow becomes [8]

$$f(x)h(x) = F(x^2, x^3). \qquad (1.8)$$

The whole point is whether the concept of single-component flow is advantageous for describing a given regular flow.

The transformation from a fixed coordinate system to a system connected with the trajectory implies, mathematically, an attempt to reduce the partial differential equation (1.2) to an ordinary differential equation. It is shown in [17] that this is a more specialized problem than the problem of separating the variables in (1.2). It is hardly surprising that the variables can be separated for the equation in a limited number of coordinate systems.

The examples of [8, 9] encouraged other authors [10–12] to look for more general (formally speaking) sufficient conditions for x^1 flow. The conditions quoted in [11] are

$$f(x) = \Phi(x^1)F(x^2, x^3) + G(x),$$
$$f(x)h(x) = \Psi(x^1)F(x^2, x^3) + H(x), \qquad (1.9)$$

where the functions G(x), H(x), $w(x^1)$ are connected by the equation

$$G(x)\frac{d^2w}{(dx^1)^2} + \frac{\partial G(x)}{\partial x^1}\frac{dw}{dx^1} + H(x)w = 0.$$

The conditions (1.4), (1.9) were shown in [17] to relate to two qualitatively different classes of flow. When (1.9) are satisfied, the order of the equation for w is lowered, with the result that the two conditions cannot be satisfied on the emitter because of this differential equation. It is obviously possible in principle to satisfy them by arranging the metric of the coordinate system in which the flow is single-component, since $\varphi = g^{11}w/2$. However, this device, like the necessary condition (1.8), has no practical value, since the coordinates involving these special properties of the metric tensor can only be determined after finding the relevant solution by some other method (that does not involve the single-component property). Hence the solutions for which (1.9) hold, and which one can hope to find by using the concept of x^1 flow, are degenerate and cannot describe the flow from an actual emitter.

The following three cases are now possible:

1) The problems of finding the coordinate system in which single-component flow is possible, and of integrating the ordinary differential equation determining this flow, are approached independently of one another. This is essentially the approach adopted in [17], and the concept of single-component flow proves useful here.

2) In addition to the solutions $W = W(x^1)$ investigated in [17], there are solutions [18] of the more general type $W = K(x^1)L(x^2)M(x^3)$; for these, the coordinate system involving single-component flow is found after solving the ordinary differential equation; this is the case for (1.7) and all other invariant solutions.

3) The coordinate system in which the flow is in the x^1 direction can be found after solving the initial partial differential equation (1.2).

The concept of x^1 flow clearly has no practical value in either the second or the third case.

§2. It will be shown that some positive results can be achieved by utilizing the necessary condition (1.8). The discussion will be confined to plane single-component flows in coordinate systems with the conformal metric

$$x^1 = \operatorname{Re} k(z), \qquad x^2 = \operatorname{Im} k(z),$$
$$g_{11} = g_{22} = \sqrt{g} \quad (z = x + iy). \qquad (2.1)$$

In this case, condition (1.8) becomes

$$\frac{1}{4g^3}\left[\left(\frac{\partial g}{\partial x^1}\right)^2 + \left(\frac{\partial g}{\partial x^2}\right)^2\right] = F(x^2). \qquad (2.2)$$

In addition to satisfying (2.2), the metric must be Euclidean. The latter implies the vanishing of the Riemann-Christoffel tensor

$$R^{p}_{\cdot rst} = 0,$$

or satisfaction of the six Lamé identities, five of which are automatic in the plane case, while the sixth becomes, recalling (2.1) [17],

$$\frac{\partial^2 g}{(\partial x^1)^2} + \frac{\partial^2 g}{(\partial x^2)^2} = \frac{1}{g}\left[\left(\frac{\partial g}{\partial x^1}\right)^2 + \left(\frac{\partial g}{\partial x^2}\right)^2\right]. \qquad (2.3)$$

1°. Consider first whether coordinate systems exist for which $g = \alpha(x^1)\beta(x^2)$ and in which x^1 flow is possible. Solution of (2.2) and (2.3) gives

$$g = a\exp(bx^2) \qquad (a, b = \text{const}). \qquad (2.4)$$

This gives the coordinates $x^1 = \psi$, $x^2 = \ln R$ and shows that flow is possible in the ψ direction. Noting that for solutions of the type considered in this section,

$$v_{x^1} = (g^{11})^{1/2}, \quad \varphi = {}^1/_2\, g^{11}, \quad \rho = \frac{1}{2\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ik}\frac{\partial g^{11}}{\partial x^k}\right),$$

the following expressions are obtained for the physical velocity component v_ψ, the potential φ, the space charge density ρ and the action W:

$$v_\psi = R^{-1}, \quad \varphi = {}^1/_2\, R^{-2}, \quad \rho = 2R^{-4}, \quad W = \psi. \qquad (2.5)$$

The existence of solution (2.5) is shown in [10]; it is written in terms of dimensionless flow parameters.

2°. Consider whether x^1 flows can exist in coordinate systems for which

$$g = [\alpha(x^1) + \beta(x^2)]^{-1}. \qquad (2.6)$$

Here, (2.2) and (2.3) give

$$\alpha'' = \alpha_0, \quad \alpha'^2 + \beta'^2 = (\alpha + \beta)(\alpha_0 + \beta''),$$
$$\alpha_0 = \text{const}. \qquad (2.7)$$

These formulas hold provided

(1) $\beta\beta'' - \beta'^2 + \alpha_0\beta = \text{const}, \quad \beta'' = \text{const}$; (2) $\alpha = \text{const}$.

It can be shown that, when α = const, the solution leads to (2.4). In case (1), we have

$$\alpha(x^1) = {}^1/_2\,\alpha_0 (x^1)^2, \quad \beta(x^2) = {}^1/_2\,\alpha_0 (x^2)^2,$$
$$g = \{{}^1/_2\alpha_0 [(x^1)^2 + (x^2)^2]\}^{-1}. \qquad (2.8)$$

The equation for g in (2.8) is the same as (1.5), apart from a constant factor. This is therefore the only solution with a metric defined by an expression of the type (2.6).

3°. Now let the determinant of the metric tensor be given by

$$g = [\alpha(x^2) + \beta(x^1)\gamma(x^2)]^{-1}. \qquad (2.9)$$

Using (2.2) and (2.3), we get

$$\beta''\gamma + \beta\gamma'' = 4F - \alpha''$$
$$(\alpha + \beta\gamma)(\beta''\gamma + \beta\gamma'' + \alpha'') = \beta'^2\gamma^2 + (\alpha' + \beta\gamma')^2. \qquad (2.10)$$

It follows at once from the first equation of (2.10) that $\gamma'' = \pm a^2\gamma$. The case when $\gamma'' = -a^2\gamma$ proves to be meaningless. Consequently,

$$\gamma = A \operatorname{ch} ax^2 + B \operatorname{sh} ax^2,$$
$$\beta = C \cos ax^1 + D \sin ax^1 + E / a^2$$
$$(A, B, C, D, E, a = \text{const}). \qquad (2.11)$$

Substituting (2.11) into the second equation of (2.10), we get the unique solution

$$\alpha(x^2) = \alpha_0 + \alpha_1 \exp(2ax^2) - \exp(ax^2),$$
$$\gamma(x^2) = \exp(ax^2)$$
$$\beta(x^1) = C \cos ax^1 + D \sin ax^1 + 1, \qquad C^2 + D^2 = 4\alpha_0\alpha_1.$$

Finally,

$$g = [\alpha_0 + \alpha_1 \exp(2ax^2) + A \cos(ax^1 + \delta) \exp(ax^2)]^{-1}. \qquad (2.12)$$

The coordinate system leading to (2.12) is given by (2.1), with

$$k(z) = a^{-1}(2i \ln \operatorname{sc} z + b). \qquad (2.13)$$

Apart from a real factor a^{-1} and a complex constant $b = \delta + i\varkappa$, (2.13) is the same as k(z) given by (1.6).

It can be shown that (2.2), (2.3) have no solutions with $g = [\alpha(x^1)\beta(x^2) + \gamma(x^1)\delta(x^2)]^{-1}$, other than (2.8) and (2.12), and no solutions with $g = \alpha(x^1)\beta(x^2) + \gamma(x^1)\delta(x^2)$.

It is clear from the above examples that condition (1.8), and the condition that the space be Euclidean, can be used effectively to establish the existence in a given class of coordinate systems of a single-component flow which does not originate from the actual emitter.

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